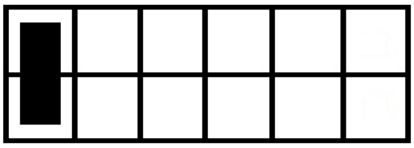
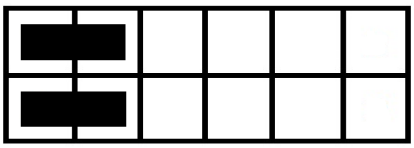
Problem 1
Tiling a band Solution

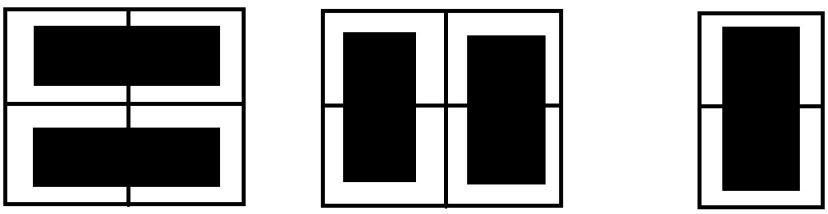
We will first take a look at the tiling of a "normal" square. Let $T(n)$ denote the number of tilings of a $2 \times n$ rectangular grid. If we start with a 2×1 block, we get that we can tile the rest of the grid in $T(N-1)$ ways.



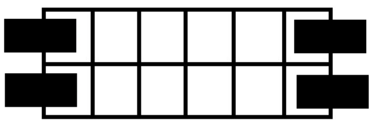
If we start with a 1×2 block, then on top there has to be another 1×2 block and we can tile the rest of the grid in $T(N-2)$ ways.



Note that none of these ways are the same. So we get the following recurrence: $T(N) = T(N-1) + T(N-2)$. Note that $T(1) = 1$ and $T(2) = 2$, as the only tilings are as follows:

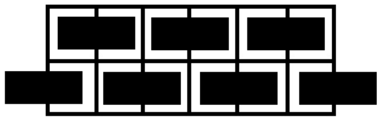


Now let's take a look at what happens if we have a band, instead of just a rectangle. Let $T'(N)$ be the number of tilings of a $2 \times n$ band. We can tile it in the same ways as we could in the rectangle case, so that are the first $T(N)$ cases. Other tilings that we are able to make on a band, but not on a rectangle, is the case in which two 1×2 blocks cross the glued side:



We see that the number of ways this is possible is $T(N-2)$.

If N is even, we have two extra cases, in which we have only 1×2 blocks, but we shift one row, so they cross the side we glued:



Therefore we get the following number of tilings on a band: $T'(N) = T(N) + T(N-2) + 2 \cdot \mathbb{1}_{\{\text{N is even}\}}$, where:

$$\mathbb{1}_{\{\text{N is even}\}} = \begin{cases} 1 & \text{if } N \text{ is even,} \\ 0 & \text{if } N \text{ is odd.} \end{cases}$$

Since $T(N) = T(N-1) + T(N-2)$, we see that $T'(N) = T(N-1) + 2T(N-2) + 2 \cdot \mathbb{1}_{\{\text{N is even}\}}$, where $T(1) = 1$ and $T(2) = 2$.